Visual representations as support for the development of multiplicative thinking

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A B S T R A C T

The development of multiplicative thinking represents one of the key steps in the mathematical education of early primary school students. The aim of this study was to examine the impact of different instructional approaches on students' achievement and their choice of strategies when solving visually represented multiplication tasks. The study was conducted on a sample of 89 second-grade primary school students, who were divided into two experimental groups and one control group. In the first experimental group, instruction was organized through a combination of visual representations and arithmetic rules, while in the second group the use of representations was linked to tasks of different semantic structures. The control group learned multiplication through an approach based on the equal groups model and repeated addition. The results indicate a statistically significant greater improvement in the experimental groups compared to the control group, as well as a higher prevalence of more efficient multiplicative strategies. In particular, rectangular arrays and decade-based representations stand out, as their use supports the development of flexible multiplication strategies.

Keywords: *multiplicative thinking, visual representations, multiplication strategies, primary school mathematics.*

Introduction

Multiplicative reasoning is a key component of primary school students' mathematical thinking, both in the early and later years of schooling (Norton et al., 2015). Research in mathematics education consistently shows that the development of multiplicative thinking is not based only on the acquisition of computational procedures and mastery of the multiplication table, but on understanding the multiple meanings of multiplication in different multiplicative situations (Greer, 1992; Siemon et al., 2005;

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Vergnaud, 1983). Although repeated addition often remains an “implicit, unconscious, and primitive model” for multiplication (Fischbein et al., 1985), scholars emphasize that the repeated-addition model is incomplete and that a significant qualitative change is needed in order to provide a basis for multiplicative thinking (Greer, 1992). Limited exposure of students to only one type of situation (e.g., equal groups) may lead to a reduced understanding of multiplication and reliance on additive strategies (Barmby et al., 2009; Kosko, 2020; Zeljic et al., 2025). In this context, visual representations play a key role because they make the structure of relationships between quantities visible and support the transition from additive to multiplicative thinking (Battista, 2003; Steffe, 1994). Research shows that the use of diverse, structurally meaningful visual representations in different multiplicative contexts supports flexibility in thinking and a deeper understanding of multiplication as an operation, not only as repeated addition (Barmby et al., 2009; Fuson, 2003; Van de Walle, 2006; Zeljic et al., 2025).

Theoretical Framework

Multiplicative thinking: from strategies to relational understanding

The application of flexible multiplication strategies is an important characteristic of multiplicative thinking. A number of authors emphasize the importance of using tasks with different semantic structures, as well as the view that the development of intuitive multiplication strategies is conditioned by the semantic structure of the problems being solved (Greer, 1992; Kouba, 1989; Mulligan & Mitchelmore, 1997; Anghileri, 1989; Downton & Sullivan, 2017).

This is also supported by the results of numerous studies showing that children possess intuitive knowledge and can solve multiplication tasks before this operation is formally introduced (Downton & Sullivan, 2017; Greer, 1992; Kouba, 1989; Mulligan & Mitchelmore, 1997). Downton and Sullivan (2017) emphasize that encouraging students to engage with more complex tasks prompts the use of sophisticated strategies. Several different classifications of strategies can be found in the literature (Table 1).

Table 1
Classification of multiplication strategies

Authors	Strategies
Mulligan & Mitchelmore (1997)	• Direct counting (counting without considering multiplicative structure) • Rhythmic counting (counting that follows the structure of the problem, e.g., 1, 2; 3, 4; 5, 6) • Skip counting (counting in groups, e.g., 2, 4, 6, …) • Additive calculation (e.g., 2 + 2 = 4, 4 + 2 = 6) • Multiplicative calculation (using learned or derived multiplication facts: 6 × 6 = 36, or if 5 × 6 = 30, then another 6 is added to 5 × 6)

Authors	Strategies
Sherin & Fuson (2005)	• "Count all" and "count by" strategies (different types of counting) • Additive calculation (repeated addition: 2 + 2 = 4, 4 + 2 = 6; as well as combining groups and adding: 4 × 8 = 16 + 16) • Pattern-based strategies (e.g., the "tens rule" or the "nine-finger rule") • Learned products (rote memorization, without visible calculation) • Hybrid strategies (combinations of the listed strategies)
Downton & Sullivan (2017)	• Transitional counting (visualization of groups using representations, e.g., fingers or drawings) • Progressive construction: double counting and counting by multiples (relying on skip counting or a combination of skip counting and doubling) • Doubling and halving (7 × 8 = 56 because 7 × 2 = 14, 14 + 14 = 28, 28 + 28 = 56) • Multiplicative calculation (according to the definition by Mulligan and Mitchelmore, 1997) • Holistic thinking (decomposing numbers using the distributive property, grouping and/or estimating using known products)

Steel and Funnell (2001) emphasize that the progression from less efficient strategies to efficient strategies does not necessarily occur if special attention is not devoted to their development. Students most often apply the strategy that they perceive as a familiar algorithm and with which they feel more certain they will succeed (Baranes et al., 1989), or because they believe that this is precisely what is expected of them. Multiplicative reasoning can also be considered through schema theory. According to this theory, the basis of multiplicative thinking is units coordination, which describes students' ability to construct and operate with different levels of composite units (units composed of units) and to apply these structures in problem solving (Hackenberg, 2010; Kosko, 2020; Steffe, 1994; Tzur et al., 2021). Three multiplicative concepts have been defined (Hackenberg, 2010; Kosko, 2020), the lowest of which involves anticipating one level of units (e.g., in the situation 4 × 6, a student can perceive four sixes), while the highest involves students being able to anticipate three levels of units (e.g., in the previous example, a student can perceive 12 × 6 as 4 × 6 + 8 × 6, but also as 10 × 6 + 2 × 6). According to this theory, additive and multiplicative reasoning do not differ in terms of the operations performed, but in terms of the schemas that the student constructs. Additive reasoning involves operating with units (counting by ones), whereas multiplicative reasoning includes the coordination of multiple levels of units – the ability to understand groups as new units and to anticipate and use them in operations (Kosko, 2020).

The role of representations in developing multiplicative thinking

Understanding multiplication does not involve only the acquisition of a new arithmetic operation, but also a qualitative change in the way students think about numbers, quantities, and units (Hiebert & Carpenter, 1992). In this context, visual representations play an important role, but only if they are structurally connected to the mathematical meaning of the operation. The literature identifies several types of iconic representations of multiplication: equal groups, rectangular arrays and area models, and linear representations/number lines (Anghileri, 2000; Barmby et al., 2009; Battista et al., 1998; Carpenter et al., 2015; Fuson, 2003; Greer, 1992; Kosko, 2020). Equal groups are one of the basic visual representations of multiplication in early instruction, in which individual units are organized into several groups with the same number of elements. These representations have an important role in the initial understanding of the basic meaning of the multiplication operation (Anghileri, 2000; Battista et al., 1998; Greer, 1992). The use of rectangular arrays is a very useful representation for the development of multiplication strategies, because it highlights the many meanings of the multiplication operation (Anghileri, 2000; Barmby et al., 2009), and not only multiplication as repeated addition (Harries & Barmby, 2007). Students who used them more successfully recognized the properties of commutativity and distributivity, which emphasizes the importance of structured visual models in explicitly representing multiplicative relationships (Jacob & Mulligan, 2014; Siemon et al., 2011; Young-Loveridge, 2005). Furthermore, the use of rectangular arrays enables students to draw rectangular arrays independently and use them as support in calculation, as well as support for the development of personal calculation strategies (Young-Loveridge, 2005). In the context of developing conceptual understanding of multiplication, base-ten visual representations have an important role in the development of multiplicative reasoning (Fuson, 2003; Van de Walle, 2006). Base-ten representations enable students to notice the base-ten structure of a number (in the context of multiplying a two-digit number) and to treat multiplication as an operation on these structured units rather than as repeated addition (Fuson, 2003; Hiebert & Wearne, 1992). For example, decomposing a product such as 7×34 into $7 \times (30 + 4)$ becomes understandable through the spatial organization of ones and tens, and the distributive law appears as a consequence of the structure of the representation, rather than as a formal rule.

The effectiveness of representations depends primarily on whether students manage to notice and coordinate the structure of the units represented by them (Barmby et al., 2009; Kosko, 2020). In other words, representations can support the understanding of multiplication only if students recognize the relationships among elements, groups, and more complex units. Research shows that the use of different visual representations can foster a deeper understanding of multiplication and the application of more flexible calculation strategies (Downton & Sullivan, 2017), and it also influences the way students conceptualize multiplicative relationships (Kosko, 2020). However, students may use different strategies even when working with the same visual representation (Kosko, 2020). This indicates that the choice of strategy does not depend exclusively on the form of the representation, but also on the way in which the structure of units within it is highlighted and interpreted. Although multiplicative structure may seem obvious to adults, numerous studies show that children do not notice this structure spontaneously (Anghileri, 1995; Barmby et al., 2009, 2011). If students' attention during instruction is not directed toward the relationships among units, visual representations may remain at the level of support for additive strategies, instead of encouraging the development of multiplicative reasoning (Lamon, 2007; Kosko, 2020). In this sense, the choice of strategies depends more on the way units are represented and quantified than on the form of the representation itself (Kosko, 2020). Accordingly, Gravemeijer (1999) describes visual representations as tools that gradually transform from contextual tools for problem solving into carriers of formal mathematical knowledge.

Aim and research questions

The preceding theoretical considerations point to two different approaches to the development of multiplicative thinking and multiplication strategies, which constituted the starting point for formulating the problem of this study. One group of authors holds that students should become familiar with arithmetic rules whose explicit application enables an understanding of multiplication strategies for single-digit numbers (Götze & Baiker, 2021; Fuson, 2003; Van de Walle, 2006). This approach, in which distributivity is explicitly addressed, encourages flexible operations with different composite units and thereby supports the development of the highest level of multiplicative thinking (Kosko, 2020; Norton et al., 2015). A second group of authors holds that understanding the properties of operations emerges from meaningful activities and problems, and not only from formal teaching (Carpenter et al., 2015; Nunes & Bryant, 1996; Siegler, 1988). According to this perspective, the introduction of arithmetic rules is not a prerequisite for the development of multiplication strategies; rather, such strategies can be encouraged through the use of different semantic structures of word problems (Ambrose et al., 2003; Anghileri, 1989; Bakker et al., 2014; Downton & Sullivan, 2017; Kouba, 1989; Mulligan & Mitchelmore, 1997; Sherin & Fuson, 2005). This study examines which of these two approaches provides a better framework for using visual representations to develop multiplication strategies. Therefore, we examine students' achievement and the use of multiplication strategies when solving tasks with different visual representations. In particular, we examine how the form and type of representation affect the understanding of a multiplicative situation and the choice of multiplication strategy.

Accordingly, the following research questions were formulated: (1) Are there differences in students' success in solving visually represented tasks depending on the instructional approach to which they were exposed? (2) Do groups of students differ in their use of task-solving strategies after the implementation of different instructional approaches? (3) In what way do the form and type of representation influence students' success in recognizing and understanding multiplicative situations, as well as their choice of multiplication strategies?

Method

A quasi-experimental design was used in the study, involving two experimental groups and one control group. The results presented in this paper are only part of the results of a broader study. The total sample consisted of 89 students from three classes of one primary school in Belgrade, Serbia. All students were 8 or 9 years old, and the classes had been balanced at school entry by the pedagogical-psychological service. One class constituted the control group (30 students), while the remaining two classes constituted the experimental groups – the Arithmetic Rules Group (31 students) and the Semantic Structures Group (28 students). The sample was convenient, since the school in which the study was conducted collaborates with the researchers' institution.

Procedure and instruments

The study was conducted in several stages: Initial test (after one lesson on the teaching unit Multiplication. The times sign.); implementation of lessons; final test. The intervention for each group was implemented over 16 lessons during a four-week period, with all lessons taught by the same teacher, who was also one of the researchers. The time for working on the tests was not limited.

For the purpose of examining success and multiplication strategies on tasks with visual representations, an initial and a final knowledge test were constructed. Each consisted of 4 tasks. These tasks assessed students' ability to: 1) compose numerical expressions that correspond to a given representation and calculate their value; 2) apply different strategies to determine the value of a product; 3) create a representation that corresponds to a given expression. Since some tasks had several items, the total number of scored items on the test was seven. The final test was the same as the initial test, with changed numbers. The tasks are presented in the appendix (Appendix 1).

Intervention

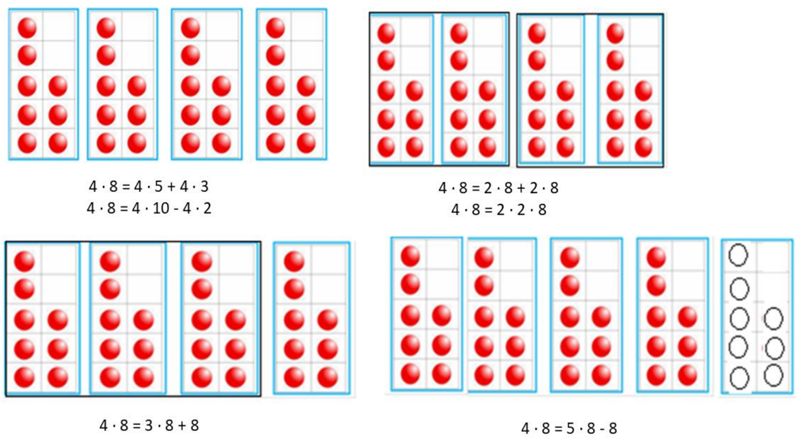
In the second-grade mathematics curriculum in the Republic of Serbia (National curriculum, 2018-2023), the outcome related to multiplication of single-digit numbers states that the student should “multiply and divide orally within the first hundred”. In the section on the implementation of the curriculum, it is stated that multiplication is introduced as repeated addition (addition of equal addends), that various examples representing a multiplicative schema are considered (when there are m elements in each of n places), and that one of the important goals is memorization of the multiplication and division tables up to 100. Although the Serbian curriculum implicitly allows the use of flexible calculation strategies by introducing appropriate arithmetic rules, the emphasis on the approach to multiplication as the sum of equal addends is reflected in methodological approaches in textbooks, in the sense that textbooks do not support the development of flexible multiplication strategies and the understanding of different multiplicative situations (Zeljić & Dabić Boričić, 2020). Therefore, our study compares achievement in multiplication and strategy use among three groups of students. The first group (Arithmetic Rules Group) was taught multiplication strategies based on arithmetic rules with the systematic use of different visual representations. The second group (Semantic Structures of Tasks Group) learned multiplication through the use of different visual representations in combination with word problems of different semantic structures. The third group, which served as the control group, received instruction based on the equal-groups model and repeated addition.

Arithmetic Rules Group

In this group, instruction on multiplication of single-digit numbers was based on the consistent application of arithmetic rules, with representations used primarily as a basis for expressing meaning and applying the rules. First, students learned multiplication by 2, 5, and 10, and then arithmetic rules (commutativity, associativity, and distributivity) were introduced using didactically adapted names and different forms of representations.

The arithmetic rules that students acquired served as the basis for introducing multiplication by the remaining single-digit numbers. Each time, students were reminded to apply the arithmetic rule they had previously learned. Students were then encouraged to use different multiplication strategies based on the representation (Figure 1), and they explained their procedures as the application of specific arithmetic rules.

Figure 1
Example of using base-ten representations



The examples were further varied by using representations of rectangular arrays (such as those mentioned above) and domino representations. At the beginning of instruction, students used visual representations as support for understanding the rules and their application, while later they performed calculations without using representations.

Semantic Structures Group

The main difference in relation to the instructional approach applied in the Arithmetic Rules Group was that students in the Semantic Structures Group were not explicitly taught arithmetic rules. Instead, they applied these rules implicitly, relying on the structures of visual representations and the semantic structures of the word problems they solved. Representations of rectangular arrays, base-ten representations, and equal groups (domino representations) were used, and calculation strategies were encouraged through their different structures and diverse grouping of composite units (Figure 1). In addition, students solved word problems with different semantic structures (equal groups, multiplicative comparison, Cartesian product, rectangular arrays, measurement relationships). Arithmetic rules were formally introduced only in the final phase of instruction on multiplication of single-digit numbers and were not used as the basis for learning multiplication, unlike in the Arithmetic Rules Group. The rules were explained through generalizations based on realistic situations, similarly to the Arithmetic Rules Group.

Control Group

In the Control Group, multiplication of single-digit numbers was learned through an additive approach. This instructional model is based on the national curriculum and supported by the textbook used by the students. Multiplication was primarily represented through the equal-groups model, while other meanings of multiplication were minimally represented. This approach integrates additive and multiplicative thinking and emphasizes the conceptual difference between situations of the type “m groups of n elements”. Determining the product of two numbers was based on repeated addition; for example, $3 \times 6 = 6 + 6 + 6$. Although the rules of commutativity and associative grouping were taught, students in this group did not learn multiplication of a sum or difference by a number (the distributive property) during instruction on multiplication of single-digit numbers. Commutativity was introduced using a rectangular array, but its application was limited to supporting multiplication through the addition of equal addends (e.g., it is easier to calculate 2×6 as $6 + 6$ than 6×2 as the sum of six twos). Similarly, associativity was introduced through a contextual example, but solely with the aim of showing that three factors can be multiplied in different orders (e.g., $2 \times (5 \times 3) = (2 \times 5) \times 3$). Although students understood these rules and could apply them to numbers, they were not used in subsequent instruction as a basis for calculation strategies, which means that their operational use was absent. The rule of multiplying a sum by a number (following the textbook as the basis for designing the operational plan) was not introduced within the topic of multiplication of single-digit numbers.

Data analysis plan

Each requirement (item) on the knowledge test (Appendix 1) was scored with 2 points if the student wrote a multiplication expression, with 1 point if the student wrote an addition expression, and with 0 points if the student did not write an expression or arrived at the answer by counting. We will present the metric characteristics of the test and then compare group achievement using one-way ANOVA for the initial test and ANCOVA for the final test. To analyze the use of strategies (the second research question), three researchers independently reviewed each test, exchanged their reports on categories, and reached consensus on the final list of categories:

– Counting (as described in Mulligan & Mitchelmore, 1997 and Sherin & Fuson, 2005);

– Repeated addition, e.g., 2 + 2 + 2 = 6 (Sherin & Fuson, 2005);

– Doubling, e.g., 7 × 8 = 56, based on doubling: double 7 is 14, then 14 + 14 = 28, and 28 + 28 = 56 (Downton & Sullivan, 2017);

– Learned products, i.e., direct recall of multiplication facts without visible calculation (Sherin & Fuson, 2005);

– Multiplicative strategies, i.e., decomposing numbers using the distributive property, associativity, and/or estimation, i.e., using known (“easy”) products to calculate the value of an unknown product (Downton & Sullivan, 2017).

To answer the question of the influence of the form and type of representation on students' success and multiplication strategies, we will analyze item difficulty indices and the distribution of strategies in the whole sample. To analyze differences in the distributions of strategy use, we used the chi-square test. The data were analyzed in SPSS.

Results

Descriptive indicators of achievement (*M* – arithmetic mean, *Md* – median, *SD* – standard deviation, Min, Max, Max% – percentage of students with the maximum number of points) for the total score on the initial and final tests are presented in Table 2.

Table 2
Descriptive indicators of achievement by group on the initial and final tests

	Control Group		Arithmetic Rules Group		Semantic Structures Group	
Test	Initial	Final	Initial	Final	Initial	Final
M	6,60	5,90	8,97	11,29	8,96	12,07
Md	6,00	6,00	9,00	14,00	10,00	14,00
SD	3,95	4,72	3,81	4,20	3,95	3,34
Min	0	0	1	0	0	2
Max	14	14	14	14	14	14
Max%	6,7%	6,7%	19,4%	54,8%	14,3%	64,3%

Scores on the initial test showed moderate reliability measured by Cronbach's alpha coefficient (α = .76), with a standard error of measurement SEM ≈ 0.95, while high reliability was recorded on the posttest (α = .90), with a lower measurement error

(SEM ≈ 1.54). A pronounced ceiling effect was observed on the final test, since a significant percentage of students in the experimental groups achieved the maximum score (14/14), indicating limited discriminativeness of the test in the upper part of the scale after the intervention and requiring caution in interpreting achievement differences on the final test. Item analysis (*r* – corrected item-total correlation, α_del – alpha coefficient if the item is deleted, *p* – difficulty index, calculated as half of the arithmetic mean) on the initial and final tests is presented in Table 3.

Table 3

Item analysis on the initial (Pre) and final test (Post)

Item	Test	1)	2a)	2b)	2v)	3)	4a)	4b)
M	Pre	1.12	1.03	0.89	0.99	1.33	1.37	1.44
	Post	1.10	1.45	1.19	1.16	1.65	1.63	1.54
SD	Pre	1.00	0.88	0.91	0.90	0.88	0.84	0.82
	Post	1.00	0.89	0.96	0.96	0.74	0.77	0.84
r	Pre	.39	.52	.51	.59	.28	.56	.57
	Post	.53	.76	.82	.82	.68	.66	.77
α_del	Pre	.76	.73	.73	.71	.78	.72	.72
	Post	.91	.88	.88	.88	.89	.90	.88
p	Pre	.560	.515	.445	.495	.665	.685	.720
	Post	.550	.725	.595	.580	.825	.815	.770

On the initial test, item difficulty indices ranged from $p = .445$ to $p = .720$, with moderate discriminativeness (corrected item-total correlation from $r = .28$ to $r = .59$). Item 3 had the lowest discriminativeness, and deleting this item would slightly increase the reliability of the scale. On the final test, an increase in the difficulty indices of most items was observed (from $p = .550$ to $p = .825$), as well as markedly better discriminative characteristics ($r = .53$ to $r = .82$), which is consistent with higher internal consistency on the final test. At the same time, the highest difficulty-index values on items 3 and 4a indicate that these items were very easy on the final test and probably contributed to the observed ceiling effect at the level of the total score.

One-way ANOVA was used to examine differences among the groups on the initial test. The assumption of homogeneity of variances was met (Levene $p = .937$). The test showed statistically significant differences in achievement on the initial test among the three groups, F (2, 86) = 3.66, $p = .030$ ($\eta p^2 \approx 0{,}08$), with post hoc comparisons (Tukey) showing that the control group had lower achievement than both experimental groups (borderline significant: $p = .052$ and $p = .060$), while the experimental groups did not differ ($p > .05$). Given the initial differences in achievement, ANCOVA was applied to compare the groups on the final test, with the final-test score as the dependent variable and the initial-test score as the covariate. The assumption of homogeneity of regression slopes was met, because the interaction between group and initial score was not statistically significant, F (2, 85) = .38, $p = .682$, $\eta p^2 = .01$. Therefore, a standard ANCOVA was conducted with the initial-test score as a covariate. The results showed a significant effect of the covariate (F (2, 85)= 13.77, $p < .001$, $\eta p^2 = .14$), indicating that initial achievement was significantly related to success on the final test. After controlling for the initial test, a statistically significant effect of group was also found (F (2, 85)= 14.56, p < .001, ηp^2 = .26), meaning that the control group and the two experimental groups differed in achievement on the final test independently of the initial level of knowledge. The control group had a significantly lower adjusted mean than both experimental groups (Table 4). Post hoc comparisons of adjusted means showed that the control group achieved statistically significantly lower results on the final test than the Arithmetic Rules Group (p < .001) and the Semantic Structures Group (p < .001). In contrast, the difference between the two experimental groups was not statistically significant (p > .05), indicating that, after controlling for initial achievement, the two implemented interventions led to comparable effects on the final test.

Table 4

Adjusted means (M_adj), standard error (SE), and 95% confidence interval (95% CI) for success on the final test

	M_adj	SE	95% CI
Control Group	6.52	.73	[5.08, 7.96]
Arithmetic Rules Group	10.97	.70	[9.58, 12.36]
Semantic Structures Group	11.76	.74	[10.29, 13.22]

Note. Calculated for the covariate value at the sample mean level, 8.17.

Frequencies and percentages of the use of different strategies on the tests are presented in Table 5. The results of the chi-square test show that the groups differed in the distribution of strategy use on the initial test $\chi^2(410, 8) = 55.06$, p < .01, CramerV = .26, and on the final test $\chi^2(407, 8) = 109.91$, p < .01, CramerV = .37.

Table 5

Frequencies and percentages of strategies on the initial (pre) and final (post) tests

Strategy	Control Group		Arithmetic Rules Group		Semantic Structures Group	
	Pre	Post	Pre	Post	Pre	Post
Counting	60	11	21	3	32	3
	48.0%	11.3%	13,6%	1.9%	24.4%	2.0%
Repeated addition	64	11	109	17	81	11
	51.2%	11.3%	70,8%	10.8%	61.8%	7.2%
Doubling	1	9	22	54	14	56
	0.8%	9.3%	14,3%	34.2%	10.7%	36.8%
Multiplication table	0	52	0	22	2	15
	0.0%	53.6%	0,0%	13.9%	1.5%	9.9%
Multiplicative strategies	0	14	2	62	2	67
	0.0%	14.4%	1,3%	39.2%	1.5%	44.1%

To reveal the influence of the form and type of representation on students' choice of strategies, we present strategy use in the total sample (Table 6). The requirement in the first task is to complete the picture (Appendix 1), so this task was not used when examining the use of strategies.

Table 6
Frequency and percentage of strategy use on each task

Strategy	Test	2a)	2b)	2v)	3)	4a)	4b)	Total
Counting	Pre	12	13	25	17	24	22	113
		10.6%	11.5%	22.1%	15.0%	21.2%	19.5%	100%
	Post	5	1	2	3	3	3	17
		29.4%	5.9%	11.8%	17.6%	17.6%	17.6%	100%
Repeated addition	Pre	47	24	37	53	45	48	254
		18.5%	9.4%	14.6%	20.9%	17.7%	18.9%	100%
	Post	1	15	8	6	5	4	39
		2.6%	38.5%	20.5%	15.4%	12.8%	10.3%	100%
Doubling	Pre	4	19	2	0	7	5	37
		10.8%	51.4%	5.4%	0.0%	18.9%	13.5%	100%
	Post	27	17	34	31	6	4	119
		22.7%	14.3%	28.6%	26.1%	5.0%	3.4%	100%
Multiplication table	Pre	0	0	0	0	1	1	2
		0.0%	0.0%	0.0%	0.0%	50.0%	50.0%	100%
	Post	33	11	4	13	16	12	89
		37.1%	12.4%	4.5%	14.6%	18.0%	13.5%	100%
Multiplicative strategies	Pre	0	0	0	1	1	2	4
		0.0%	0.0%	0.0%	25.0%	25.0%	50.0%	100%
	Post	4	12	7	26	45	49	143
		2.8%	8.4%	4.9%	18.2%	31.5%	34.3%	100%

Discussion

The results of the study show that different instructional approaches to multiplication affect students' success in solving visually represented multiplicative tasks, that is, their understanding of different multiplicative representations. After controlling for initial achievement, the analysis of covariance showed that students in both experimental groups achieved statistically significantly better results on the final test than the control group, while the difference between the two experimental groups was not statistically significant. Therefore, regardless of whether students were taught using arithmetic rules or semantic structures, systematic work with different representations of multiplication contributed to students' greater success in solving visually represented multiplicative tasks. These results confirm the view that instruction focused exclusively on the equal-groups model does not enable students to develop the breadth of meanings of multiplication (Anghileri, 2000; Barmby et al., 2009; Gravemeijer, 1999; Hackenberg, 2010; Kosko, 2020; Steffe, 1994; Tzur et al., 2021), but rather encourages a limited understanding of multiplication as repeated addition. Within such a conceptual framework, students often fail to recognize multiplicative structure in new contexts or visual representations, which may lead to lower achievement on tasks that require more flexible understanding (Kosko, 2020; Lamon, 2007).

A particularly important finding concerns the dynamics of achievement within the control group. Descriptive indicators show that students in the control group did not make progress during the intervention period; on the contrary, their average result on the final test was somewhat lower than on the initial test (Table 2). This result indicates not only the absence of a positive effect of the applied approach, but also possible stagnation or regression in the understanding of multiplicative relationships. The results of the control group show insufficiently developed understanding of multiplicative situations, which may also be a consequence of the overemphasis on additive thinking in instruction (Clark & Kamii, 1996; Greer, 1992), from which students find it difficult to move away if it is dominantly represented. These results also confirm that students do not spontaneously notice multiplicative structure (Anghileri, 1995; Barmby et al., 2009; Barmby et al., 2011). On the other hand, exposing students to different contexts enables them to form a network of connected meanings and to recognize the common structure in different representations of the same situation (Downton & Sullivan, 2017; Fuson, 2003; Siemon et al., 2005; Van de Walle, 2006). This finding supports the view that visual models function as carriers of the meaning of the multiplication operation, because they enable the coordination of multiple levels of units and the linking of symbolic notation with the meaning of the representation of a multiplicative situation (Anghileri, 2000; Hiebert & Carpenter, 1992; Kosko, 2020; Mulligan & Mitchelmore, 1997).

The second research task concerned the examination of the influence of the way in which multiplication of single-digit numbers is taught on the development of flexible multiplication strategies. The results of the study indicate significant changes in students' multiplication strategies between the initial and final testing. On the initial test, counting and repeated addition strategies dominated in all groups, indicating limited multiplicative understanding and reliance on the conception of multiplication as “shortened” addition. This finding is consistent with the view that the development of multiplicative thinking represents a gradual transition from additive to relational understanding of relationships between quantities (Greer, 1992; Siemon et al., 2005). On the final test, a significant decrease in the use of these strategies and an increase in the use of multiplication-table knowledge and more flexible multiplication strategies were observed. Thus, regardless of the teaching model, students will manage to master multiplication of single-digit numbers to a similar extent. However, Kilpatrick (2001) emphasized the importance of learning procedures and calculation strategies with understanding, because acquiring them without understanding leads to difficulties in calculations with multi-digit numbers (Zeljic et al., 2025). Students in the control group did not show a strong increase in flexible multiplicative strategies. The control group most often relied on learned products to determine the product of two numbers (53.6% of the strategies used), significantly more than the Arithmetic Rules Group (13.9%) and the Semantic Structures Group (9.9%). Students in the control group also often used repeated addition and counting (11.3% each). The dominant use of factual knowledge and the absence of efficient strategies indicate that students learned multiplication facts by heart and that their thinking remained additive in nature (Baroody, 2006; Steel & Funnell, 2001; Zeljic et al., 2019; Zeljic et al., 2025). In other words, instruction based on equal groups and repeated addition enables the use of counting, repeated addition, and memorization of products, but does not encourage the transition to higher levels of unit coordination and, consequently, to higher levels of multiplicative reasoning (Hackenberg, 2010; Steffe, 1994). Regardless of the efficiency of a strategy, students resorted to the strategies in which they felt most secure (Baranes et al., 1989). Although such knowledge enables efficient task solving, it does not necessarily imply understanding different multiplicative situations or the ability to choose a strategy depending on the context (Greer, 1992; Siemon et al., 2005). Our results confirm the view that, without systematic support through different representations and activities, students less frequently develop more sophisticated forms of multiplicative reasoning spontaneously (Steel & Funnell, 2001). By contrast, in the experimental groups a pronounced increase was observed in doubling strategies and multiplicative strategies, with less reliance on purely memorized multiplication-table knowledge. The results show that the use of different representations also encourages the application of holistic strategies (Battista, 2003; Downton & Sullivan, 2017; Hackenberg, 2010; Jacob & Mulligan, 2014; Kosko, 2020; Siemon et al., 2011; Steffe, 1994; Tzur et al., 2021), as well as shaping the way students conceptualize multiplicative relationships (Kosko, 2020). Work with diverse representations can encourage the construction of multiplication strategies without direct reliance on formal arithmetic rules, which is consistent with studies showing that strategies can be developed through meaningful activities and problem solving (Downton & Sullivan, 2017; Mulligan & Mitchelmore, 1997). The results also show that younger children are capable of understanding the properties of operations when these are grounded in meaningful activities and problems (Carpenter et al., 2015; Nunes & Bryant, 1996), as well as applying them as a basis for flexible strategies (Götze & Baiker, 2021; Fuson, 2003; Van de Walle, 2006).

The next research question concerned the influence of the form of representation on the development of multiplicative thinking and multiplication strategies. The results of the study show that the form and type of representation have a significant influence on both students' success and their choice of multiplication strategies, which is contrary to the results reported by Kosko (2020). The first task on the test required students to complete a partially started rectangular array on the basis of a given expression. Research shows that the ability to construct a rectangular representation independently is an indicator of more developed multiplicative thinking (Gravemeijer, 1999; Mulligan & Mitchelmore, 1997; Young-Loveridge, 2005), which enables the application of different strategies. The value of the difficulty index for this task indicates moderate success, both on the initial test (p = .560) and on the final test (p = .550). This result points to the need for further work on developing students' multiplicative thinking. The relatively low success may be partly explained by students' limited experience, especially in the control group, with rectangular arrays in instruction.

The second task was based on the equal-groups model, but the same situation was represented through different structuring of composite units: as 6 × 4, but also as 2 × 4 + 2 × 4 + 2 × 4 and 3 × 4 + 3 × 4. The greatest difficulties occur in tasks 2b and 2v (difficulty indices on the initial test p = .515 and p = .445, respectively), where the structure is represented as the sum of partial products. Students must notice that several composite units can represent the same quantity as a single product, which requires a higher level of conceptual understanding. According to the theory of units coordination, the ability to construct and coordinate composite units represents the basis of multiplicative thinking (Hackenberg, 2010; Kosko, 2020; Steffe, 1994). These results indicate that students often experience a multiplicative situation as a fixed structure rather than as a relationship that can be reorganized (Lamon, 2007; Kosko, 2020). On the third task with a rectangular array, students showed better achievement (p = .825), especially on the final test. This supports the views that the use of rectangular arrays supports understanding of commutativity and distributivity and encourages the use of flexible multiplication strategies (Anghileri, 2000; Harries & Barmby, 2007; Young-Loveridge, 2005). In terms of schema theory, they facilitate the transition toward higher multiplicative concepts. The greatest success on the initial test was recorded on tasks with base-ten representations (the fourth task, p = .685 and p = .720) and success on this task was also high on the final test (p = .815 and p = .770, respectively). The spatial organization of tens and ones enables students to view multiplication as an operation on structured units, rather than as repeated addition (Fuson, 2003; Hiebert & Wearne, 1992; Kosko, 2020; Tzur et al., 2021).

The largest number of strategies that students used intuitively before systematic instruction in multiplication were counting and repeated addition (113 and 254 strategies, respectively). A significant increase in the doubling strategy in task 2b on the initial test (51.4% of the use of the doubling strategy occurred on this task) indicates the development of the ability to work with more complex units (Hackenberg, 2010; Kosko, 2020). On the other hand, on the final test there was a noticeable increase in the use of multiplicative strategies on tasks with rectangular and base-ten representations – of the 143 multiplicative strategies used, 84% were used on the third and fourth tasks. These results partially confirm the view that the choice of strategies depends more on the way the unit is represented and quantified than on the form of the representation itself (Kosko, 2020).

Among the limitations of the study, it may be noted that the research was conducted on a convenience sample of students from one school institution, with the results interpreted within the instructional context and the current curriculum. In addition, the number of tasks for individual types of visual representations was limited, which was conditioned by the scope of the broader study, while this paper presents selected parts of the overall research material.

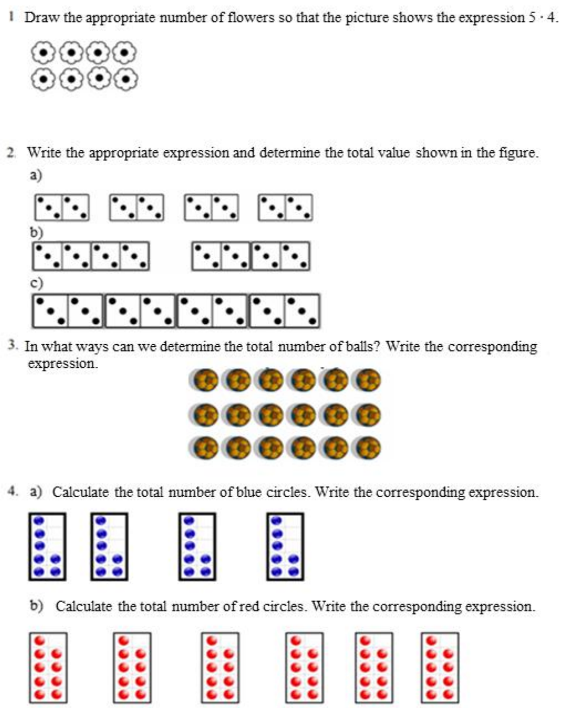
Conclusion

The results of the study show that students' success in solving visually represented multiplication tasks depends to a significant extent on the instructional model that was applied, with the form of representation also emerging as an important factor. The Control Group, which learned multiplication primarily through the equal-groups model and repeated addition, did not show progress on the final test; rather, it achieved somewhat lower results than on the initial measurement. The formal introduction of arithmetic rules in instruction will not lead to their operational application without the use of representations and problems that are carriers of the meaning of rules and calculation procedures. Our results show that the additive approach, although important in the early phase and dominantly emphasized in our curriculum and textbooks, is not sufficient for the development of multiplicative thinking.

Overall, the results of this study have important implications for the design of multiplication instruction in early mathematics education. They indicate that the development of multiplicative thinking requires instruction that goes beyond the repeated-addition model and systematically includes work with different representations, semantic structures of tasks, and arithmetic rules as carriers of mathematical meaning. The results particularly point to the need for greater representation of base-ten representations in instructional practice and textbook materials, since they support the understanding of multiplication as an operation on structured units and encourage the development of more flexible multiplicative strategies. Nevertheless, in practice they are often insufficiently represented in early multiplication instruction, as well as in studies examining the role of representations in understanding multiplication of single-digit numbers.

Appendix

Appendix 1
Tasks used in the research



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